Preprints of the 18th IFAC World Congress
Milano (Italy) August 28 - September 2, 2011

Robust Adaptive Fuzzy Sliding Mode Control Design for a class of MIMO underactuated system

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Abstract: In this paper, a robust adaptive fuzzy controller which combines the sliding mode control with an adaptive Proportional Integral (PI) term is developed for a class of nonlinear Multi Inputs Multi Outputs (MIMO) underactuated systems with unknown parameters and in presence of external disturbances. The main contribution of the proposed method is that underactuated systems can be controlled in their original non square form. The problem of matrix singularity for this class of systems is solved by using the property of the regularized inverse and by introducing a compensator term in the proposed control law. The free parameters of the adaptive fuzzy controller are tuned on-line based on the Lyapunov approach. The overall adaptive fuzzy scheme guarantees the boundedness of all the closed-loop signals as well as the tracking errors. The validity of the proposed approach is shown by computer simulations.

Keywords: Sliding mode fuzzy control, underactuated systems, robustness.

1. INTRODUCTION

In the recent years, research works have been done on adaptive fuzzy sliding mode control of multivariable non linear perturbed systems. Nonetheless, the chattering phenomenon represents a limitation for this technique. Then, this problem has been addressed (Utkin et al. (2000), Aloui et al. (2010c), Roopaei et al. (2009), Hamzaoui et al. (2003) and their references). However, most of the recent research works have considered MIMO nonlinear systems where the number of inputs and outputs is equal. Recently, there has been a growing attention in underactuated systems characterized by the fact that they have fewer actuators than the degrees of freedom to be controlled Santhakumar and Asokan (2009), Aloui et al. (2010a), Aloui et al. (2010b), Kung et al. (2009), Yi and Wang. (2005), Hao et al. (2008). This class of mechanical systems are abundant in reality for instance, space craft, underwater robots, helcopters, mobile robots...In literature, several works have proposed different techniques to study the difficult control problem of underactuated system. In Hao et al. (2008), a robust control using incremental sliding mode for underactuated systems with mismatched uncertainties is developed. In Kung et al. (2009), the authors have proposed an adaptive fuzzy sliding mode controller for a class of underactuated systems. In fact, a fourth order underactuated system is decoupled into two subsystems and a sliding surface for each subsystem is defined. In Santhakumar and Asokan (2009), an adaptive non linear control strategy for underactuated autonomous underwater vehicle is presented. In Yi and Wang. (2005), a cascade sliding mode control approach is used to treat a large class of underactuated systems. In Aloui et al. (2010b), a stable adaptive fuzzy sliding mode based tracking control is developed for a class of non-square nonlinear uncertain systems. In order to eliminate the chattering phenomenon brought by the conventional variable structure control, the signum function is replaced by an adaptive Proportional Derivative (PD) term in the proposed approach.
In this paper, we intend to study a fuzzy adaptive control law for a class of underactuated nonlinear systems with external disturbances in their original non square form, Aloui et al. (2010a), Aloui et al. (2010b). A robust control algorithm is developed by combining the adaptive fuzzy sliding mode approach with an adaptive PI term in order to simultaneously to reduce the chattering phenomenon as well as to ensure the good tracking performances though the presence of the external disturbances and all approximation errors. On the basis of the Lyapunov theory, the proposed controller is shown to guarantee the H_∞ tracking performance.
The remainder of this paper is organized as follows. The problem under investigation is first formulated in section *2*. The controller design and the main results are described in Section *3*. The simulation studies are given in Section *4* to validate the proposed scheme. Section *5* gives a conclusion on the main works developed in this paper.

2. CONVENTIONNEL SLIDING MODE FOR UNDERACTUATED SYSTEM

Consider the following MIMO nonlinear underactuated system given by:

$$\begin{aligned} y_1^{(r_1)} &= f_1(x) + g_{11}(x)u_1 + \cdots + g_{1p}(x)u_p + d_{1_{r_1}} \\ \cdots &= \cdots \\ y_m^{(r_m)} &= f_m(x) + g_{m1}(x)u_1 + \cdots + g_{mp}(x)u_p + d_{m_{rm}} \end{aligned} \quad (1)$$

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where $x = [\ y_1\ \cdots\ y_1^{r_1-1}\ \cdots\ y_m\ \cdots\ y_m^{r_m-1}\ ]^T$ is the state vector,
$U = [\ u_1\ \cdots\ u_p\ ] \in \mathbb{R}^p$ is the control input vector,
$Y = [\ y_1\ \cdots\ y_m\ ] \in \mathbb{R}^m$ is the output vector and note also that: $p < m$. $f_i\ g_{ij} \in \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $j = 1\ \cdots\ p\ \ i = 1 \cdots m$, are smooth partially unknown function vectors, $D = [\ d_1\ \cdots\ d_m\ ]^T$ is the perturbation vector where $d_i = [\ 0\ 0\ \cdots\ d_{i_{r_i}}\ ]$, $i = 1\ \cdots\ m$ (where X^T is the transpose of X).

Assumption 1. The external disturbance d_i is assumed to belong to $L_2[\ 0\ t\ ]$, $\forall t \in [0,\infty)$ and $|\ d_i\ | < D_i\ \forall i$. $D_i\ \forall i$ is supposed to be unknown.

Denote

$$Y(t) = \begin{bmatrix} y_1^{(r_1)} \\ \cdots \\ y_m^{(r_m)} \end{bmatrix}, F(x) = \begin{bmatrix} f_1(x) \\ \cdots \\ f_m(x) \end{bmatrix}, U = \begin{bmatrix} u_1 \\ \cdots \\ u_p \end{bmatrix}\ d = \begin{bmatrix} d_{1_{r_1}} \\ \cdots \\ d_{m_{r_m}} \end{bmatrix}$$

$G(x) = [G_1\ G_2 \cdots G_m]^T$ where $G_i = [g_{i1}\ g_{i2} \cdots g_{ip}]^T\ i = 1, \cdots, m$
Then (1) can be written as:

$$Y = F(x) + G(x)U + d \tag{2}$$

Definition 2.1. Let $G(x)$ be a $m \times p$ matrix, and $G^+(x)$ be the pseudo-inverse of $G(x)$. If $G(x)$ is of full rank, then $G(x)^+$ can be computed as follows:

$$G^+(x) = \left[G(x)^T G(x)\right]^{-1} G^T(x), \qquad m > p \tag{3}$$

The output tracking error e_i is:

$$e_i = y_i - y_{d_i} \tag{4}$$

y_{d_i} is the desired reference trajectory of y_i. Let us consider desired trajectories $y_{d_i}(t)$, $i = 1 \cdots m$ that are known bounded functions of time with bounded known derivatives and are assumed to be *ri-time* differentiable.
The sliding mode surfaces are defined in the space of the tracking errors by the following equation:

$$s_i = \sum_{j=1}^{r_i} \alpha^i_{(r_i-j)} \left(\frac{d}{dt}\right)^{(r_i-j)} e_i \tag{5}$$

with $\alpha^i_{(r_i-1)} = 1$.
The surface vector is defined by:

$$S = [s_1 \cdots s_m]^T \tag{6}$$

The parameters $\alpha^i_{(r_i-2)}, \cdots, \alpha^i_0$ are chosen such that all roots $h_i(s) = s^{(r_i-1)} + \alpha^i_{(r_i-2)} s^{(r_i-2)} + \cdots + \alpha^i_1 s + \alpha^i_0$ are in the left half of the complexe plane. (here s denotes the complex Laplace transform variable).
The control law is designed as:

$$U = u_{eq} + u_{sw} \tag{7}$$

The equivalent control law u_{eq} is determined by $\dot{S} = 0$.
From (2), (5) and (6), the time derivative of S can be obtained as

$$\dot{S} = F(x) + G(x)U + d - Y_d^{(n_r)} + \sum_{i=2}^{n_r} \Lambda_{(i-2)} E^{(i-1)}$$

Where:

$$n_r = \max_i \{r_i\}$$

$$Y_d^{(n_r)} = \left[y_{d_i}^{(r_1)} y_{d_i}^{(r_2)} \cdots y_{d_i}^{(r_m)}\right]^T\ i = 1, 2, \cdots m$$

$$\Lambda_i = diag\left[\alpha_i^1 \text{ or } \emptyset, \alpha_i^2 \text{ or } \emptyset, \cdots, \alpha_i^m \text{ or } \emptyset\right]\ i = 0, 1, \cdots, n_r - 2$$

$$E^{(i)} = \left[e_1^{(i)} e_2^{(i)} \cdots e_m^{(i)}\right]^T\ i = 1, 2, \cdots n_r - 1$$

For underactuated systems, in order to avoid the singularity problem of $G(x)G^+(x)$, we consider in this case the following approximation Labiod et al. (2005), since we need $\left(G(x)G^+(x)\right)^{-1}$ in the equivalent control term:

$$\left(G(x)G^+(x)\right)^{-1} \simeq \left(G(x)G^+(x)\right)^T \left[\varepsilon_0 I + \left(G(x)G^+(x)\right)\left(G(x)G^+(x)\right)^T\right]^{-1} \tag{8}$$

where I is the identity matrix of appropriate dimension. ε_0 is small positive constant.
The equivalent control term is defined by the following expression

$$u_{eq} = G^+(x)\left(G(x)G^+(x)\right)^T \left(\varepsilon_0 I + \left(G(x)G^+(x)\right)\left(G(x)G^+(x)\right)^T\right)^{-1} \left(-F(x) + Y_d^{(n_r)} - \sum_{i=2}^{n_r} \Lambda_{(i-2)} E^{(i-1)} + u_r\right) \tag{9}$$

We introduce another term u_r in the control law in order to cancel the approximation error from (8) and to ensure the stability of the system (more details are given next).
Denote $\left(G(x)G^+(x)\right) = \underline{G}(x)$
u_r satisfies the following expression in order to get $\dot{S} = 0$ and after some manipulations:

$$u_r = \left[\varepsilon_0 I + \underline{G}(x)\underline{G}(x)^T\right]\underline{G}(x)\left[\varepsilon_0 I + \underline{G}(x)^T\underline{G}(x)\right]^{-1}\underline{G}(x)^T\left[\varepsilon_0 I + \underline{G}(x)\underline{G}(x)^T\right]^{-1}\varepsilon_0\left[\varepsilon_0 I + \underline{G}(x)\underline{G}(x)^T\right]^{-1}\left(-\hat{F}(\hat{x} \mid \theta_1) + Y_d^{(n_r)} - \sum_{i=2}^{n_r} \Lambda_{(i-2)} E^{(i-1)}\right) \tag{10}$$

The switching control term is defined by:

$$u_{sw} = -G^{-1}(x)\eta\, sgn(S) \text{ with } \eta > D_i \quad \forall i \tag{11}$$

sgn is the signum function
The sliding mode control law is given by:

$$U = G^+(x)\underline{G}(x)^T\left(\varepsilon_0 I_m + \underline{G}(x)\underline{G}(x)^T\right)^{-1}\left(-F(x) + Y_d^{(n_r)} - \sum_{i=2}^{n_r} \Lambda_{(i-2)} E^{(i-1)} + u_r - \eta\, sgn(S)\right) \tag{12}$$

It is obvious that in order to obtain the sliding mode control, the system functions $F(x)$ and $G(x)$ and the switching parameters η have to be known.
In the next section, we will use adaptive fuzzy systems to approximate nonlinear functions F and G. Then, in order to eliminate the chattering phenomenon without deteriorating the tracking performance, the discontinuous term in the conventional sliding mode technique is replaced by an adaptive PI term.

3. DESIGN OF A ROBUST ADAPTIVE FUZZY CONTROLLER

Control objectives: Determine a robust adaptive control law $U(x)$ for the nonlinear system given by (1) such that the vector

y can follow a given desired reference signal vector y_d, under that the following conditions are met:
(i) The closed-loop system is stable, i.e., all the signals involved are uniformly bounded.
(ii) For a given disturbance attenuation level $0 < \rho < 1$, the following H_∞ tracking performance is achieved as:

$$\int_0^t E^T Q E dt \leq E(0)^T P E(0) + \frac{1}{\gamma_f}\tilde{\theta}_1^T(0)\tilde{\theta}_1(0) + \frac{1}{\gamma_g}\tilde{\theta}_2^T(0)\tilde{\theta}_2(0) + \rho^2 \int_0^t W'^T W' dt$$

for all $W \in L_2[0,t], \forall t \in [0,\infty)$, where $Q = Q^T \geq 0$ and $P = P^T \geq 0$ are symmetric positive definite weighting matrices, $\tilde{\theta}_1 = \theta_1^* - \theta_1$ and $\tilde{\theta}_2 = \theta_2^* - \theta_2$ are parameters approximation errors defined next and, γ_f, γ_g are the adaptation gains. W' gather all the terms relating to the approximation errors and the external disturbances. We introduce the notation $W' \in L_2[\,0\ t\,]$ $t \in \mathbb{R}^+ \Longleftrightarrow$

$$\int_0^t W'^T W' dt < \infty \quad \forall t \in \mathbb{R}^+ \tag{13}$$

Step 1: Approximation $F(x)$ and $G(x)$

Both functions $F(x)$ and $G(x)$ in (2) are unknown, that is why we design two fuzzy systems $\hat{F}(x \mid \theta_1)$ and $\hat{G}(x \mid \theta_2)$ to approximate them.
Let $A_i^k, k = 1, \cdots, n_i$ be the fuzzy sets defined on the universe of discourse of the ith input i.e. x_i. The fuzzy system is characterized by a set of if-then rules in the following form:

$$\text{Rule}^{\,k}: \text{ If } x_1 \text{ is } A_1^k \text{ and } \cdots \text{ and } x_n \text{ is } A_n^k \text{ then } z \text{ is } G^k \quad (k = 1, \cdots N) \tag{14}$$

where $A_i^k \in \{A_i^1, \cdots, A_i^{n_i}\}$ $i = 1, \cdots, n$, G^k are fuzzy sets defined respectively for x_i $i = 1, \cdots, n$ and z and $N = \prod_{i=1}^{n} n_i$ is the total number of rules.
z is given by:

$$z = \theta^T \xi(x) \tag{15}$$

where $\theta^T = [\theta_1, \theta_2, \cdots, \theta_N]$ is a vector grouping all consequent parameters and $\xi = [\xi_1, \xi_2, \cdots, \xi_N]^T$ is a set of fuzzy basis functions defined as

$$\xi_k(x) = \frac{\prod_{i=1}^{n} \mu_{A_i^k}(x_i)}{\Sigma_{i=1}^{n}\left(\prod_{i=1}^{n} \mu_{A_i^k}(x_i)\right)} \tag{16}$$

where $\mu_{A_i^k}(x_i)$ is the membership function and represents the fuzzy meaning of the symbol A_i^k. θ_k is the value of the singleton associated with G^k.
The output of $\hat{F}(x \mid \theta_1)$ and $\hat{G}(x \mid \theta_2)$ can be respectively obtained by:

$$\hat{F}(x \mid \theta_1) = \left[\hat{f}_1(x|\theta_{f_1}) \cdots \hat{f}_m(x|\theta_{f_m})\right]^T = \theta_1^T \xi_f(x) \tag{17}$$

Where $\hat{f}_j(x|\theta_{f_j}) = \theta_{f_j}^T \xi_f(x)$ $j = 1 \cdots m$ and $\theta_1^T = \begin{bmatrix} \theta_{f_1}^T \\ \cdots \\ \theta_{f_m}^T \end{bmatrix}$

$$\hat{G}(x \mid \theta_2) = \begin{bmatrix} \hat{g}_{11}(x|\theta_{g_{11}}) & \cdots & \hat{g}_{1p}(x|\theta_{g_{1p}}) \\ \cdots & \cdots & \cdots \\ \hat{g}_{m1}(x|\theta_{g_{m1}}) & \cdots & \hat{g}_{mp}(x|\theta_{g_{mp}}) \end{bmatrix} = \theta_2^T \Phi(x) \tag{18}$$

$\hat{g}_{ij}(x|\theta_{g_{ij}}) = \theta_{g_{ij}}^T \xi_g(x)$ $i = 1 \cdots p$, $j = 1 \cdots m$.

$$\theta_2^T = \begin{bmatrix} \theta_{g_{11}}^T & \cdots & \theta_{g_{1p}}^T \\ \vdots & \ddots & \vdots \\ \theta_{g_{m1}}^T & \cdots & \theta_{g_{mp}}^T \end{bmatrix}; \Phi(x) = \begin{bmatrix} \xi_g(x) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \xi_g(x) \end{bmatrix}$$

Step 2: Reducing chattering phenomenon by adding a PI term

The presence of the signum function in the term u_{sw} in the classical sliding mode technique leads to the chattering phenomenon, which can excite the high frequency dynamics. To avoid this problem and to achieve the previous control objectives, an adaptive PI term is used. In fact, the integral term when added to the proportional term accelerates the movement of the process towards the set point and eliminates the residual steady-state error that occurs with a proportional term. Hence, the inputs and outputs of the continuous time PI term are of the following form:

$$u_{PI} = \begin{bmatrix} u_{PI_1} \\ \cdots \\ u_{PI_m} \end{bmatrix} = \begin{bmatrix} k_{P_1} s_1 + k_{i_1} \int_{t_0}^{t} s_1(\tau) d\tau \\ \cdots \\ k_{P_m} s_m + k_{i_m} \int_{t_0}^{t} s_m(\tau) d\tau \end{bmatrix} \tag{19}$$

Where k_{P_j} and k_{i_j} $j = 1 \cdots m$ are the control gains to be computed. (19) can be rewritten as:

$$\begin{aligned} u_{PI} = \hat{\rho}(S \mid \theta_p) &= \left[\left(\hat{\rho}_1(s_1 \mid \theta_{\rho_1}) \ldots \left(\hat{\rho_m}(s_m \mid \theta_{\rho_m})\right)\right]^T \\ &= \left[\theta_{\rho_1}^T \Theta(s_1) \ldots \theta_{\rho_m}^T \Theta(s_m)\right]^T = \Theta(S)\theta_\rho \end{aligned} \tag{20}$$

Where adjustable parameters vector θ_{ρ_j} are given by $\theta_{\rho_j} = [k_{p_j}, k_{i_j}]^T$
and $\Theta^T(s_j) = [s_j, \int_{t_0}^{t} s_j(\tau) d\tau]$ are regressive vectors, $j = 1 \cdots m$,
$\theta_\rho = (\theta_{\rho_1}^T \cdots \theta_{\rho_m}^T)^T$,
$\Theta(S) = diag[\Theta^T(s_1) \cdots \Theta^T(s_m)]$.

Step 3: Control law design

The control law is obtained:

$$\begin{aligned} U = \hat{G}^+(x \mid \theta_2)\left(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\right)^T \Big(\varepsilon_0 I + \Big(\hat{G}(x \mid \theta_2) \\ \hat{G}^+(x \mid \theta_2)\Big)\left(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\right)^T\Big)^{-1}\Big(-\hat{F}(x \mid \theta_1) \\ + Y_d^{(n_r)} - \sum_{i=2}^{n_r} \Lambda_{(i-2)} E^{(i-1)} + u_r - u_{PI} + \hat{u}_{rob}\Big) \end{aligned} \tag{21}$$

The term $\hat{u}_{rob}$ in (21) detailed next is the robust compensator defined as:

$$\hat{u}_{rob} = -\frac{2}{\rho} S \tag{22}$$

The term u_r is also to be designed next.

Theorem 1. Consider the class of MIMO nonlinear systems (1), the control law (21) where the functions $\hat{F}(x|\theta_1)$, $\hat{G}(x|\theta_2)$ and u_{PI} are given by (17), (18) and (20), the parameter vectors θ_1, θ_2, θ_ρ are adjusted by the adaptive laws (23). Suppose that the assumptions *2*, *3*, 4 and *5* are satisfied, then (21) ensures that all the closed-loop signals are bounded and the tracking errors decrease asymptotically to zero.
Adaptive laws:

$$\begin{aligned}\dot{\theta}_1 &= \gamma_f \xi_f(x) S^T \\ \dot{\theta}_2 &= \gamma_g \Phi(x) U S^T \\ \dot{\theta}_\rho &= \gamma_\rho \Theta(S)^T S\end{aligned} \tag{23}$$

Proof
First, let us define the following variables:

$$\theta_{\rho_i}^* = \underset{\theta_{\rho_i}\in\Omega_{\rho_i}}{arg\ min}\Big(\sup_{S\in R^m} \|\hat{\rho}_i(S \mid \theta_{p_i}) - \eta sgn(s_i)\|\Big) \tag{24}$$

Where $\eta sgn(s_i)$ is the discontinuous term of the conventional sliding mode control and Ω_{ρ_i} denotes the set of the suitable bounds on θ_{ρ_i}.

$$\text{and } \Omega_{\rho_i} = \Big\{\theta_{\rho_i} \in \Re^2 / \|\theta_{\rho_i}\| \le M_{\rho_i}\Big\} \tag{25}$$

Denote:

$$w_{i_{PI}} = -\hat{\rho}(S \mid \theta_{\rho_i}^*) + \eta sgn(s_i) \tag{26}$$

$$W_{PI} = [w_{1_{PI}} \cdots w_{m_{PI}}]^T \tag{27}$$

Assumption 2. The approximation error $w_{i_{PI}}(t)$ is assumed to belong to $L_2[0,t]$, $\forall t \in [0,\infty)$.

Define also the optimal parameters vector θ_1^*, θ_2^* of the previous fuzzy systems:

$$\theta_1^* = \underset{\theta_1\in\Omega_{\theta_1}}{arg\ min}\Big(\sup_{x\in\Omega_x} \|F(x) - \hat{F}(x \mid \theta_1)\|\Big) \tag{28}$$

$$\theta_2^* = \underset{\theta_2\in\Omega_{\theta_2}}{arg\ min}\Big(\sup_{x\in\Omega_x} \|G(x) - \hat{G}(x \mid \theta_2)\|\Big) \tag{29}$$

Where $\Omega_{\theta_1}, \Omega_{\theta_2}$ denote the sets of suitable bounds on θ_1, θ_2 respectively and Ω_x denotes the set of the suitable bounds on x. Assume that the constraint sets Ω_{θ_1} are specified as:

$$\Omega_{\theta_1} = \Big\{\theta_1 / \|\theta_1\| \le M_f\Big\} \tag{30}$$

$$\text{and } \Omega_{\theta_2} = \Big\{\theta_2 / \|\theta_2\| \le M_g\Big\} \tag{31}$$

Note while assuming that the fuzzy parameter vectors θ_1, θ_2 and x never reach the boundary of Ω_{θ_1}, Ω_{θ_2} and Ω_x respectively, we can define the vector of the minimum approximation errors as:

$$W_{Fuzzy} = [w_{1_F} \cdots w_{m_F}]^T \tag{32}$$

with

$$w_{i_F} = f_i(x) - \hat{f}(x \mid \theta_{f_i}^*) + \sum_{j=1}^{p}(g_{ij}(x) - \hat{g}_{ij}(x \mid \theta_{g_ij}^*))u_j \ \ i = 1,\cdots,m \tag{33}$$

Assumption 3. The approximation error $w_{i_F}(t)$ is assumed to belong to $L_2[0,t]$, $\forall t \in [0,\infty)$.

The time derivative of S can be obtained from (20) and (21)

$$\dot{S} = F(x) + d - Y_d^{(n_r)} + \sum_{i=2}^{n_r}\Lambda_{(i-2)}E^{(i-1)} + \Big(G(x) - \hat{G}(x \mid \theta_2)\Big)U$$

$$\dot{S} = F(x) + d - Y_d^{(n_r)} + \sum_{i=2}^{n_r}\Lambda_{(i-2)}E^{(i-1)} + \Big(G(x) - \hat{G}(x \mid \theta_2) + \hat{G}(x \mid \theta_2)\Big)U \tag{34}$$

by using the following property:

$$\Big[\varepsilon I + \Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)\Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)^T\Big]\Big[\varepsilon_0 I + \Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)\Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)^T\Big]^{-1} = I$$

(34) can be rewritten as follows:

$$\begin{aligned}\dot{S} = {} & F(x) - \hat{F}(x \mid \theta_1) + \Big(G(x) - \hat{G}(x \mid \theta_2)\Big)U - \hat{\rho}(S \mid \theta_p) \\ & - \hat{u}_{rob} + d + \Big(I - \varepsilon_0\Big[\varepsilon_0 I + \Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big) \\ & \Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)^T\Big]^{-1}\Big)u_r - \varepsilon_0\Big[\varepsilon_0 I + \Big(\hat{G}(x \mid \theta_2) \\ & \hat{G}^+(x \mid \theta_2)\Big)\Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)^T\Big]^{-1}\Big)\Big(-\hat{F}(x \mid \theta_1) \\ & + Y_d^{(n_r)} - \sum_{i=2}^{n_r}\Lambda_{(i-2)}E^{(i-1)} - \hat{\rho}(S \mid \theta_p) - \hat{u}_{rob}\Big)\end{aligned} \tag{35}$$

When admitted that u_r satisfies the following equation:

$$\begin{aligned}u_r = {} & \Big[\varepsilon_0 I + \Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)\Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)^T\Big] \\ & \Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)\Big[\varepsilon_0 I + \Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)^T \\ & \Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)\Big]^{-1}\Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)^T\Big[\varepsilon_0 I \\ & + \Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)\Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)^T\Big]^{-1} \\ & \varepsilon_0\Big[\varepsilon_0 I + \Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)\Big(\hat{G}(x \mid \theta_2)\hat{G}^+(x \mid \theta_2)\Big)^T\Big]^{-1} \\ & \Big(-\hat{F}(x \mid \theta_1) + Y_d^{(n_r)} - \sum_{i=2}^{n_r}\Lambda_{(i-2)}E^{(i-1)} + \hat{\rho}(S \mid \theta_p) - \hat{u}_{rob}\Big)\end{aligned} \tag{36}$$

(35) can be rewritten from (24), (28),(29) and (36):

$$\begin{aligned}\dot{S} = {} & \hat{F}(\hat{x} \mid \theta_1^*) - \hat{F}(x \mid \theta_1) + \big[\hat{G}(\hat{x} \mid \theta_2^*) - \hat{G}(x \mid \theta_2)\big]U \\ & + \hat{\rho}(S \mid \theta_{\rho_i}^*) - \hat{\rho}(S \mid \theta_p) - \hat{\rho}(S \mid \theta_{\rho_i}^*) - \frac{2}{\rho}S + W_{Fuzzy} + d \\ \dot{S} = {} & \tilde{\theta}_1^T\xi_f(x) + \tilde{\theta}_2^T\Phi(x)U + \Theta(S)\tilde{\theta}_\rho - \frac{2}{\rho}S - \eta sgn(S) \\ & + W_{PI} + W_{Fuzzy} + d \\ \dot{S} = {} & \tilde{\theta}_1^T\xi_f(x) + \tilde{\theta}_2^T\Phi(x)U + \Theta(S)\tilde{\theta}_\rho - \frac{2}{\rho}S - \eta sgn(S) + W'\end{aligned}$$

Where $sgn(S) = [sgn(s_1)\cdots sgn(s_m)]^T$, $\tilde{\theta}_1 = \theta_1^* - \theta_1$, $\tilde{\theta}_2 = \theta_2^* - \theta_2$, $\tilde{\theta}_\rho = \theta_\rho^* - \theta_\rho$ and $W' = W_{Fuzzy} + W_{PI} + d$. Then, consider a Lyapunov function candidate as follows:

$$V = \frac{1}{2}tr(S^TS) + \frac{1}{2\gamma_f}tr(\tilde{\theta}_1^T\tilde{\theta}_1) + \frac{1}{2\gamma_g}tr(\tilde{\theta}_2^T\tilde{\theta}_2) + \frac{1}{2\gamma_\rho}tr(\tilde{\theta}_\rho^T\tilde{\theta}_\rho) \tag{37}$$

where $\gamma_f > 0, \gamma_g > 0$ and $\gamma_\rho > 0$ are the adaptation gains. Then, the time derivative of V is given by:

$$\begin{aligned}\dot{V} = {} & \frac{1}{\gamma_f}tr\Big[\tilde{\theta}_1^T(\gamma_f\xi_f(x)S^T + \dot{\tilde{\theta}}_1)\Big] + \frac{1}{\gamma_\rho}tr\Big[(\gamma_\rho S^T\Theta(S) + \dot{\tilde{\theta}}_\rho^T)\tilde{\theta}_\rho\Big] \\ & + \frac{1}{\gamma_g}tr\Big[\tilde{\theta}_2^T(\gamma_\rho\Phi(x)US^T + \dot{\tilde{\theta}}_2)\Big] - \eta tr(sgn(S)S^T) \\ & - \frac{2}{\rho}tr(SS^T) + tr(W'S^T)\end{aligned} \tag{38}$$

Then:

$$\dot{\tilde{\theta}}_1 = -\dot{\theta}_1,\ \dot{\tilde{\theta}}_2 = -\dot{\theta}_2 \text{ and } \dot{\tilde{\theta}}_\rho = -\dot{\theta}_\rho \tag{39}$$

Denote: $|S| = |s_1| + |s_2| + \cdots + |s_m|$, substituting the adaptive laws (23) and (39) in (38), we obtain:

$$\dot{V} \leq -\eta \mid S \mid -\frac{2}{\rho}S^T S + S^T W'$$

$$\dot{V} \leq -\Big[\frac{1}{\sqrt{\rho}}S - \frac{\sqrt{\rho}}{2}W'\Big]^T\Big[\frac{1}{\sqrt{\rho}}S - \frac{\sqrt{\rho}}{2}W'\Big] - \frac{2}{\rho}S^T S + \frac{1}{\rho}S^T S + \frac{1}{4}\rho W'^T W'$$

$$\text{Hence, } \dot{V} \leq -\frac{1}{\rho}S^T S + \frac{1}{4}\rho W'^T W' \tag{40}$$

Integrating (40) for $t = 0$ to T_r:

$$V(T_r) - V(0) \leq -\frac{1}{\rho}\int_0^{T_r} S^T S\, dt + \frac{1}{4}\rho\int_0^{T_r} W'^T W'\, dt \tag{41}$$

Then since $V(T) > 0$ and $0 < \rho < 1$, (41) can be rewritten as:

$$\int_0^{T_r} S^T S\, dt \leq 2V(0) + \rho^2\int_0^{T_r} W'^T W'\, dt \tag{42}$$

Using Barbalat's lemma Wang. (1993), we can see that the sliding surfaces asymptotically converges to zero in finite time despite the external disturbances.
Furthermore, it can be proved from (42) that the H_∞ tracking performance in (13) is achieved, that guarantees the global stability and the robustness of the closed loop system.
The global stability is achieved under the assumption that all the parameter vectors are within the constraint sets. To guarantee that the parameters are bounded, the adaptive laws (24) can be modified by using the projection algorithm **?**.

4. SIMULATION RESULTS

In this section, the performance of the proposed approach is evaluated for an inverted pendulum system to illustrate the H_∞ tracking performance of the proposed control design algorithm. The structure of an inverted pendulum described by the following dynamic equations Kung et al. (2009).

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1 + b_1 u + d_1 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2 + b_2 u + d_2 \\ y_1 = x_1 \\ y_2 = x_3 \end{cases} \tag{43}$$

Where:
$x_1 = \theta$ is the angle of the pole with respect to the vertical axis;
$x_2 = \dot{\theta}$ is the angular velocity of the pole with respect to the vertical axis;
$x_3 = x$ is the position of the cart;
$x_4 = \dot{x}$ is the velocity of the cart;
u is the applied force to move the cart

$$f_1 = \frac{m_t g \sin x_1 - m_p L \sin x_1 \cos x_1 x_2^2}{L(\frac{4}{3}m_t - m_p \cos^2 x_1)}$$

$$b_1 = \frac{\cos x_1}{L(\frac{4}{3}m_t - m_p \cos^2 x_1)}$$

$$f_2 = \frac{-\frac{4}{3}m_p L x_2^2 \sin x_1 + m_p g \sin x_1 \cos x_1}{\frac{4}{3}m_t - m_p \cos^2 x_1}$$

$$b_2 = \frac{4}{3(\frac{4}{3}m_t - m_p \cos^2 x_1)}$$

g : is the acceleration of gravity
L : is the angular velocity of the pole with respect to the vertical axis;
m_c : is the cart mass;
m_p : is the pole mass;
Define $m_t = m_p + m_c$
The control objective is to force the system outputs y_1 and y_2 to track the desired trajectories with the same frequency $y_{d_1}(t) = \pi/30 sin(t)$ and $y_{d_2}(t) = sin(t)$ respectively in spite of the disturbances. Figure. 1-7 show the simulation results under the proposed adaptive fuzzy sliding mode algorithm. The tracking curves of states, its reference signals are displayed in Figure 1-4 respectively, which indicate the good performance of our controller in particularly the good tracking of the reference signals: $y_{d_1}, \dot{y}_{d_1}, y_{d_2}, \dot{y}_{d_2}$ as well as the robustness of the control system with respect to the disturbance. In Figure 5, it is shown that there is no chattering in the control law. The evolution of the surfaces trajectories are shown in Figure 6-7 and we see the convergence to zero of the system and the attractiveness of the sliding surfaces.

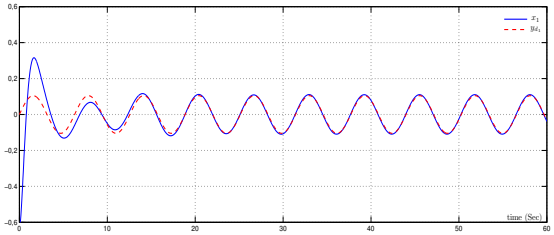


Fig. 1. Trajectories of the angle of the pole (solide line) and the reference signal y_{d_1} (dashed line)(rad)

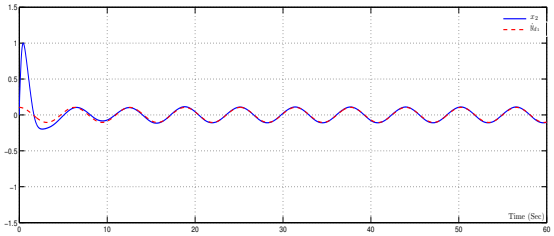


Fig. 2. Trajectories of the angular velocity of the pole (solide line) and the reference signal $\dot{y}_{d_1}$(dashed line)(rad/s)

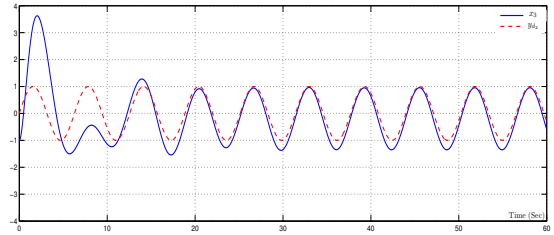


Fig. 3. Trajectories of the position of the cart (solide line) and the reference signal y_{d_2} (dashed line)(m)

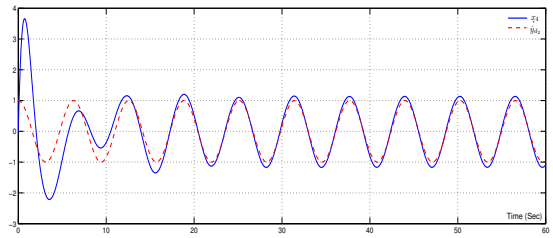


Fig. 4. Trajectories of the velocity of the cart (solide line) and their reference signal $\dot{y}_{d_2}$ (dashed line)(m/s)

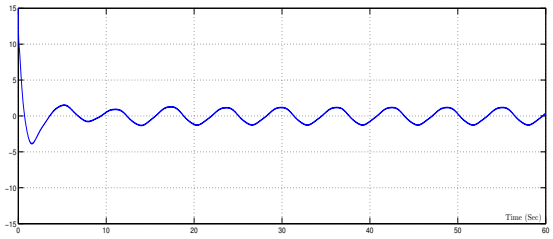


Fig. 5. Control input signal $N.m$

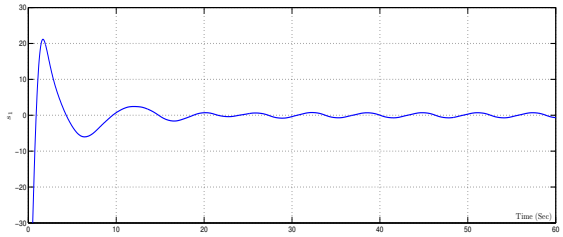


Fig. 6. Evolution of surface s_1

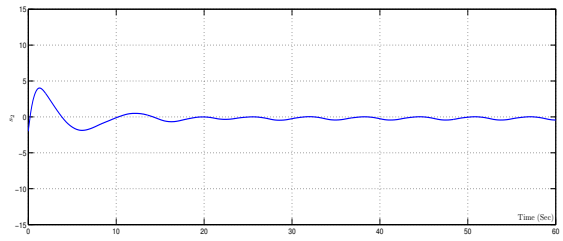


Fig. 7. Evolution of surface s_2

5. CONCLUSION

In this paper, an adaptive fuzzy controller using sliding mode approach is developed for a class of underactuated nonlinear disturbed systems. The main contribution of our proposed method in comparison with other methods is the relaxation of the condition that the system should be controlled in their square form (by adding or eliminating variables). Due to the fact that the pseudo inverse of the gain matrix is used, the introduced approximation error is taken into account in the control law. To obtain the control objectives (stability and robustness), the fuzzy approximator is used to approximate unknown nonlinear functions. In addition, this work has proposed to combine a sliding mode control and an adaptive PI term into a single fuzzy sliding mode controller in order to overcome the chattering problem and to ensure the tracking of desired trajectories. Finally, based on the Lyapunov stability approach, the proposed adaptive fuzzy sliding mode control scheme can guarantee the robustness of the closed loop system with respect to disturbance as well as the tracking of desired trajectories. Nonlinear system simulation results are presented to verify the effectiveness of the proposed method.

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